

Global well-posedness, scattering and blow-up for the energy-critical, focusing Hartree equation in the radial case

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Abstract

We establish global existence, scattering for radial solutions to the energy-critical focusing Hartree equation with energy and \dot{H}^1 norm less than those of the ground state in $\mathbb{R} \times \mathbb{R}^d$, $d \geq 5$.

Key Words: Focusing Hartree equation, Global well-posedness, Scattering, Long time perturbation.

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1 Introduction

We consider the following initial value problem

$$\begin{cases} iu_t + \Delta u = f(u), & \text{in } \mathbb{R}^d \times \mathbb{R}, \quad d \geq 5, \\ u(0) = u_0(x), & \text{in } \mathbb{R}^d, \end{cases} \quad (1.1)$$

where $u(t, x)$ is a complex-valued function in spacetime $\mathbb{R} \times \mathbb{R}^d$ and Δ is the Laplacian in \mathbb{R}^d , $f(u) = -(|x|^{-4} * |u|^2)u$. It is introduced as a classical model in [31]. In practice, we use the integral formulation of (1.1)

$$u(t) = U(t)u_0(x) - i \int_0^t U(t-s)f(u(s))ds, \quad (1.2)$$

where $U(t) = e^{it\Delta}$.

We are primarily interested in (1.1) since it is critical with respect to the energy norm. That is, the scaling $u \mapsto u_\lambda$ where

$$u_\lambda(t, x) = \lambda^{\frac{d-2}{2}}u(\lambda^2t, \lambda x), \quad \lambda > 0 \quad (1.3)$$

maps a solution to (1.1) to another solution to (1.1), and u and u_λ have the same energy (2.2).

It is known that if the initial data $u_0(x)$ has finite energy, then (1.1) is locally well-posed (see, for instance [23]). That is, there exists a unique local-in-time solution that lies in $C_t^0 \dot{H}_x^1 \cap L_t^6 L_x^{\frac{6d}{3d-8}}$ and the map from the initial data to the solution is locally Lipschitz in these norms. If the energy is small, it is known that the solution exists globally in time and scattering occurs; That is, there exist solutions u_\pm of the free Schrödinger equation $(i\partial_t + \Delta)u_\pm = 0$ such that

$$\|u(t) - u_\pm(t)\|_{\dot{H}_x^1} \rightarrow 0 \quad \text{as } t \rightarrow \pm\infty.$$

However, for initial data with large energy, the local well-posedness argument do not extend to give global well-posedness, only with the conservation of the energy (2.2), because the time of existence given by the local theory depends on the profile of the data as well as on $\|u_0\|_{\dot{H}_x^1}$.

A large amount of work has been devoted to the theory of scattering for the Hartree equation, see [4]-[9], [22]-[25], [27] and [28]. In particular, global well-posedness in \dot{H}_x^1 for the energy-critical, defocusing Hartree equation in the case of large finite-energy initial data was obtained recently by us [24], [25]. In this paper, we continue this investigation and establish scattering result for radial solutions to the energy-critical, focusing Hartree equation for data with energy and \dot{H}^1 norm less than those of the ground state.

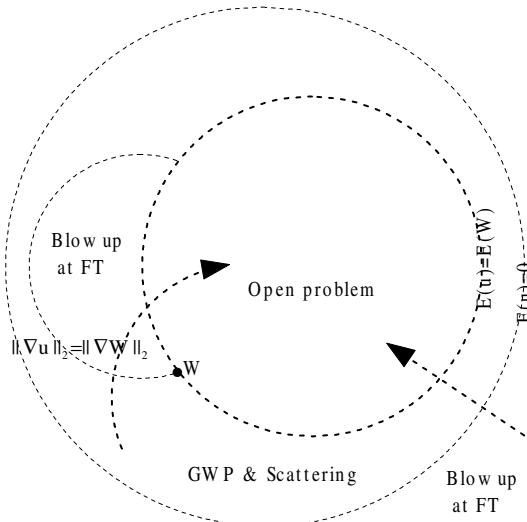


Figure 1: A description of the solutions with radial data in the energy space, where “FT” denotes finite time.

The main result of this paper is the following global well-posedness and blow up results for (1.1) in the energy space (Figure 1).

Theorem 1.1. *Let $d \geq 5$, $u_0 \in \dot{H}^1(\mathbb{R}^d)$ be radial and let u be the corresponding solution*

to (1.1) in $\dot{H}^1(\mathbb{R}^d)$ with maximal forward time interval of existence $[0, T)$. Suppose $E(u_0) < E(W)$.

- (1) If $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then $T = +\infty$ and u scatters in \dot{H}^1 .
- (2) If $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$, then $T < +\infty$, and thus, the solution blows up at finite time.

Similar as in [12], it is still open that scattering for the general data with energy and \dot{H}^1 norm less than those of the ground state. But concerning the blow up result, we also have

Theorem 1.2. *Let $d \geq 5$, $u_0 \in \dot{H}^1(\mathbb{R}^d)$ and let u be the corresponding solution to (1.1) in $\dot{H}^1(\mathbb{R}^d)$ with maximal forward time interval of existence $[0, T)$. Suppose $E(u_0) < E(W)$, $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$ and $|x|u_0 \in L^2$, then $T < +\infty$, i.e., the solution blows up at finite time.*

Next, we introduce some notations. If X, Y are nonnegative quantities, we use $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $X \leq CY$ for some C which may depend on the critical energy E_{crit} (see Section 4) but not on any parameter such as η , and $X \approx Y$ to denote the estimate $X \lesssim Y \lesssim X$. We use $X \ll Y$ to mean $X \leq cY$ for some small constant c which is again allowed to depend on E_{crit} .

We use $C \gg 1$ to denote various large finite constants. and $0 < c \ll 1$ to denote various small constants.

The Fourier transform on \mathbb{R}^d is defined by

$$\widehat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx,$$

giving rise to the fractional differentiation operators $|\nabla|^s$, defined by

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \widehat{f}(\xi).$$

These define the homogeneous Sobolev norms

$$\|f\|_{\dot{H}_x^s} := \| |\nabla|^s f \|_{L_x^2(\mathbb{R}^d)}.$$

Let $e^{it\Delta}$ be the free Schrödinger propagator. In physical space this is given by the formula

$$e^{it\Delta} f(x) = \frac{1}{(4\pi it)^2} \int_{\mathbb{R}^d} e^{\frac{i|x-y|^2}{4t}} f(y) dy,$$

while in frequency space one can write this as

$$\widehat{e^{it\Delta} f}(\xi) = e^{-it|\xi|^2} \widehat{f}(\xi).$$

In particular, the propagator preserves the above Sobolev norms and obeys the dispersive estimate

$$\|e^{it\Delta} f\|_{L_x^\infty(\mathbb{R}^d)} \lesssim |t|^{-\frac{d}{2}} \|f\|_{L_x^1(\mathbb{R}^d)}, \quad \forall t \neq 0. \quad (1.4)$$

Let $d \geq 5$, a pair (q, r) is L^2 -admissible if

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right), \text{ for } 2 \leq r \leq \frac{2d}{d-2}.$$

For a spacetime slab $I \times \mathbb{R}^d$, we define the *Strichartz* norm $\dot{S}^0(I)$ by

$$\|u\|_{\dot{S}^0(I)} := \sup_{(q,r) \text{ L}^2\text{-admissible}} \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)}.$$

and for some fixed number $0 < \epsilon_0 \ll 1$, define $\mathcal{Z}^1(I)$ by

$$\|u\|_{\mathcal{Z}^1(I)} := \sup_{(q,r) \in \wedge} \|u\|_{L_t^q L_x^r},$$

where

$$\wedge = \left\{ (q, r); \frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right) - 1, \frac{2d}{d-2} \leq r \leq \frac{2d}{d-4} - \epsilon_0 \right\}.$$

When $d \geq 5$, the spaces $(\dot{S}^0(I), \|\cdot\|_{\dot{S}^0(I)})$ and $(\mathcal{Z}^1(I), \|\cdot\|_{\mathcal{Z}^1(I)})$ are Banach spaces, respectively.

We will occasionally use subscripts to denote spatial derivatives and will use the summation convention over repeated indices.

We work in the frame of [12], [13] and [16]. In Section 2, we recall some useful facts. In Section 3, we obtain some variational estimates and blow up results (Part (2) of Theorem 1.1 and Theorem 1.2). Last using a concentration compactness argument, we obtain the scattering result (Part (1) of Theorem 1.1) in Section 4 and 5.

2 A review of the Cauchy problem

In this section, we will recall some basic facts about the Cauchy problem

$$\begin{cases} iu_t + \Delta u = f(u), & (x, t) \in \mathbb{R}^d \times \mathbb{R}, \quad d \geq 5, \\ u(t_0) \in \dot{H}^1(\mathbb{R}^d), \end{cases} \quad (2.1)$$

where $f(u) = -(|x|^{-4} * |u|^2)u$. It is the \dot{H}^1 critical, focusing Hartree equation.

Based on the above notations, we have the following *Strichartz* inequalities

Lemma 2.1 (Strichartz estimate[11], [30]). *Let u be an \dot{S}^0 solution to the Schrödinger equation (2.1). Then*

$$\|u\|_{\dot{S}^0} \lesssim \|u(t_0)\|_{L_x^2} + \|f(u)\|_{L_t^{q'} L_x^{r'}(I \times \mathbb{R}^d)}$$

for any $t_0 \in I$ and any admissible pairs (q, r) . The implicit constant is independent of the choice of interval I .

From Sobolev embedding, we have

Lemma 2.2. *For any function u on $I \times \mathbb{R}^d$, we have*

$$\|\nabla u\|_{L_t^\infty L_x^2} + \|\nabla u\|_{L_t^6 L_x^{\frac{6d}{3d-2}}} + \|\nabla u\|_{L_t^3 L_x^{\frac{6d}{3d-4}}} + \|u\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} + \|u\|_{L_t^6 L_x^{\frac{6d}{3d-8}}} \lesssim \|\nabla u\|_{\dot{S}^0},$$

where all spacetime norms are on $I \times \mathbb{R}^d$.

For convenience, we introduce two abbreviated notations. For a time interval I , we denote

$$\|u\|_{X(I)} := \|u\|_{L_t^6(I; L_x^{\frac{6d}{3d-8}})}; \quad \|u\|_{Y(I)} := \|\nabla u\|_{L_t^6(I; L_x^{\frac{6d}{3d-2}})}; \quad \|u\|_{W(I)} := \|\nabla u\|_{L_t^3(I; L_x^{\frac{6d}{3d-4}})}.$$

We develop a local well-posedness and blow-up criterion for the \dot{H}^1 -critical Hartree equation. First, we have

Proposition 2.1 (Local well-posedness [24]). *Suppose $\|u(t_0)\|_{\dot{H}^1} \leq A$, I be a compact time interval that contains t_0 such that*

$$\|U(t - t_0)u(t_0)\|_{X(I)} \leq \delta,$$

for a sufficiently small absolute constant $\delta = \delta(A) > 0$. Then there exists a unique solution $u \in C_t^0 \dot{H}_x^1$ to (2.1) on $I \times \mathbb{R}^d$, such that

$$\|u\|_{W(I)} < \infty, \quad \|u\|_{X(I)} \leq 2\delta.$$

Moreover, if $u_{0,k} \rightarrow u_0$ in $\dot{H}^1(\mathbb{R}^d)$, the corresponding solutions $u_k \rightarrow u$ in $C(I; \dot{H}^1(\mathbb{R}^d))$.

Remark 2.1. *There exists $\tilde{\delta} > 0$, such that if $\|u(t_0)\|_{\dot{H}^1} \leq \tilde{\delta}$, the conclusion of Proposition 2.1 applies to any interval I . In fact, by Strichartz estimates, we have*

$$\|e^{i(t-t_0)\Delta} u(t_0)\|_{X(I)} \leq C \|e^{i(t-t_0)\Delta} u(t_0)\|_{Y(I)} \leq C \tilde{\delta},$$

and the claim follows.

Remark 2.2. *Given $u_0 \in \dot{H}^1$, there exists I such that $0 \in I$ and the hypothesis of Proposition 2.1 is satisfied on I . In fact, by Strichartz estimates, we have*

$$\|e^{it\Delta} u_0\|_{Y(I)} < \infty,$$

then the claim follows from Sobolev inequality and absolutely continuity theorem.

Remark 2.3 (Energy identity). *Based on the standard limiting argument, if u is the solution constructed in Proposition 2.1, we have that*

$$E(u(t)) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 - \frac{1}{4} \iint \frac{1}{|x-y|^4} |u(t,x)|^2 |u(t,y)|^2 \, dx dy. \quad (2.2)$$

is constant for $t \in I$.

Now let $t_0 \in I$. We say that $u \in C(I; \dot{H}^1(\mathbb{R}^d)) \cap W(I)$ is a solution of (2.1) if

$$u|_{t_0} = u_0, \quad \text{and} \quad u(t) = e^{i(t-t_0)\Delta} u_0 - i \int_{t_0}^t e^{i(t-s)\Delta} f(u) ds$$

with $f(u) = -(|x|^{-4} * |u|^2)u$. Note that if $u^{(1)}, u^{(2)}$ are solutions of (2.1) on I , $u^{(1)}(t_0) = u^{(2)}(t_0)$, then $u^{(1)} \equiv u^{(2)}$ on $I \times \mathbb{R}^d$. This is because we can partition I into a finite collection of subintervals I_j with

$$A = \sup_{t \in I} \max_{i=1,2} \|u^{(i)}(t)\|_{\dot{H}^1}.$$

If j_0 is such that $t_0 \in I_{j_0}$, then the uniqueness of the fixed point in the proof of Proposition 2.1, combined with Remark 2.2 gives an interval $\tilde{I} \ni t_0$ so that $u^{(1)}(t) = u^{(2)}(t), t \in \tilde{I}$. A continuation argument now easily gives $u^{(1)}(t) = u^{(2)}(t), t \in I$.

Definition 2.1 (Maximal interval). *The above analysis allows us to define a maximal interval $(t_0 - T_-(u_0), t_0 + T_+(u_0))$, with $T_\pm(u_0) > 0$, where the solution is defined. If $T_1 < t_0 + T_+(u_0)$, $T_2 > t_0 - T_-(u_0)$, $T_2 < t_0 < T_1$, then u solves (2.1) in $[T_2, T_1] \times \mathbb{R}^d$, so that $u \in C([T_2, T_1], \dot{H}^1(\mathbb{R}^d)) \cap X([T_2, T_1]) \cap W([T_2, T_1])$.*

Proposition 2.2 (Blow-up criterion [24]). *If $T_+(u_0) < +\infty$, then*

$$\|u\|_{X(t_0, t_0 + T_+(u_0))} = +\infty.$$

A corresponding result holds for $T_-(u_0)$.

Definition 2.2 (Nonlinear profile). *Let $v_0 \in \dot{H}^1$, $v(t) = e^{it\Delta} v_0$ and let t_n be a sequence, with $\lim_{n \rightarrow \infty} t_n = \bar{t} \in [-\infty, \infty]$. We say that $u(t, x)$ is a nonlinear profile associated with $(v_0, \{t_n\})$ if there exists an interval I , with $\bar{t} \in I$ (if $\bar{t} = \pm\infty$, $I = [a, +\infty)$ or $(-\infty, a]$) such that u is a solution of (2.1) in I and*

$$\lim_{n \rightarrow \infty} \|u(t_n, \cdot) - v(t_n, \cdot)\|_{\dot{H}^1} = 0.$$

Remark 2.4. *Similar as in [12], there always exists a unique nonlinear profile $u(t)$ associated to $(v_0, \{t_n\})$, with a maximal interval I .*

Last, in order to meet our needs in Lemma 4.2, we give a stability theory, which is somewhat different from that in [25], but their proofs are similar in essence.

Proposition 2.3 (Long-time perturbations). *Let I be a compact interval, and let \tilde{u} be a function on $I \times \mathbb{R}^d$ which obeys the bounds*

$$\|\tilde{u}\|_{X(I)} \leq M \quad (2.3)$$

and

$$\|\tilde{u}\|_{L_t^\infty(I; \dot{H}_x^1)} \leq E \quad (2.4)$$

for some $M, E > 0$. Suppose also that \tilde{u} is a near-solution to (2.1) in the sense that it solves

$$(i\partial_t + \Delta)\tilde{u} = -(|x|^{-4} * |\tilde{u}|^2)\tilde{u} + e \quad (2.5)$$

for some function e . Let $t_0 \in I$, and let $u(t_0)$ be close to $\tilde{u}(t_0)$ in the sense that

$$\|u(t_0) - \tilde{u}(t_0)\|_{\dot{H}_x^1} \leq E'$$

for some $E' > 0$. Assume also that we have the smallness conditions

$$\|e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\|_{\mathcal{Z}^1(I)} \leq \epsilon, \quad (2.6)$$

$$\|e\|_{L_t^{\frac{3}{2}}(I; \dot{H}_x^{1, \frac{6d}{3d+4}})} \leq \epsilon \quad (2.7)$$

for some $0 < \epsilon < \epsilon_1$, where ϵ_1 is some constant $\epsilon_1 = \epsilon_1(E, E', M) > 0$.

We conclude that there exists a solution u to (2.1) on $I \times \mathbb{R}^d$ with the specified initial data $u(t_0)$ at t_0 , and

$$\|u\|_{\mathcal{Z}^1(I)} \leq C(M, E, E').$$

Moreover, we have

$$\|\nabla u\|_{S^0(I)} \leq C(M, E, E').$$

Remark 2.5. Under the assumptions (2.3) and (2.7), we know that the assumption (2.4) is equivalent to the following condition

$$\|\nabla \tilde{u}(t_0)\|_{L^2} \leq E.$$

Remark 2.6. The long time perturbation theorem in [25] yields the following continuity fact, which will be used later: Let $\tilde{u}_0 \in \dot{H}^1$, $\|\tilde{u}_0\|_{\dot{H}^1} \leq A$, and let \tilde{u} be the solution of (2.1), with maximal interval of existence $(T_-(\tilde{u}_0), T_+(\tilde{u}_0))$. Let $u_{0,n} \rightarrow \tilde{u}_0$ in \dot{H}^1 , and let u_n be the corresponding solution of (2.1), with maximal interval of existence $(T_-(u_{0,n}), T_+(u_{0,n}))$. Then

$$\begin{aligned} T_-(\tilde{u}_0) &\geq \overline{\lim}_{n \rightarrow +\infty} T_-(u_{0,n}), \\ T_+(\tilde{u}_0) &\leq \underline{\lim}_{n \rightarrow +\infty} T_+(u_{0,n}), \end{aligned}$$

and for each $t \in (T_-(\tilde{u}_0), T_+(\tilde{u}_0))$, $u_n(t) \rightarrow \tilde{u}(t)$ in \dot{H}^1 .

3 Some variational estimates and blow-up result

Let $W(x)$ be the ground state to be the positive radial Schwartz solution to the elliptic equation

$$\Delta W + (|x|^{-4} * |W|^2)W = 0. \quad (3.1)$$

The existence and uniqueness of W were established in [17] and [20]. By invariance of the equation, for $\theta_0 \in [-\pi, \pi]$, $\lambda_0 > 0$, $x_0 \in \mathbb{R}^d$,

$$W_{\theta_0, x_0, \lambda_0}(x) = \lambda_0^{-\frac{d-2}{2}} e^{i\theta_0} W\left(\frac{x - x_0}{\lambda_0}\right)$$

is still a solution. Now let C_d be the best constant of the Sobolev inequality in dimension d . That is,

$$\forall u \in \dot{H}^1, \quad \|(|x|^{-4} * |u|^2)|u|^2\|_{L^1}^{\frac{1}{4}} \leq C_d \|\nabla u\|_{L^2}. \quad (3.2)$$

In addition, using the concentration-compactness argument [10], [18], [19] and [26], we can obtain the following characterization of W :

If $\|(|x|^{-4} * |u|^2)|u|^2\|_{L^1}^{\frac{1}{4}} = C_d \|\nabla u\|_{L^2}$, $u \neq 0$, then $\exists(\theta_0, \lambda_0, x_0)$ such that $u = W_{\theta_0, x_0, \lambda_0}$.

From above, we have

$$\|(|x|^{-4} * |W|^2)|W|^2\|_{L^1} = C_d^4 \left(\int |\nabla W|^2 dx \right)^2.$$

On the other hand, from (3.1), we obtain

$$\|(|x|^{-4} * |W|^2)|W|^2\|_{L^1} = \int |\nabla W|^2 dx.$$

Hence, we have

$$\|\nabla W\|_{L^2}^2 = \frac{1}{C_d^4}, \quad E(W) = \left(\frac{1}{2} - \frac{1}{4}\right) \|\nabla W\|_{L^2}^2 = \frac{1}{4C_d^4}.$$

Lemma 3.1. *Assume that*

$$\|\nabla u\|_{L^2} < \|\nabla W\|_{L^2}.$$

Assume moreover that $E(u) \leq (1 - \delta_0)E(W)$ where $\delta_0 > 0$. Then, there exists $\bar{\delta} = \delta_0^{1/2} > 0$ such that

$$\begin{aligned} \int |\nabla u|^2 dx - \iint \frac{|u(x)|^2 |u(y)|^2}{|x - y|^4} dx dy &\geq \frac{\bar{\delta}}{2} \int |\nabla u|^2 dx, \\ \int |\nabla u|^2 dx &\leq (1 - \bar{\delta}) \int |\nabla W|^2 dx, \\ E(u) &\geq 0. \end{aligned}$$

Proof: Define

$$a = \int |\nabla u|^2 dx \quad \text{and} \quad f(x) = \frac{1}{2}x - \frac{1}{4}C_d^4 x^2.$$

From (3.2), we have

$$(1 - \delta_0)E(W) \geq E(u) \geq \frac{1}{2} \int |\nabla u|^2 dx - \frac{1}{4}C_d^4 \left(\int |\nabla u|^2 dx \right)^2 = f(a). \quad (3.3)$$

Note that

$$f'(x) = \frac{1}{2} - \frac{1}{2}C_d^4 x,$$

This implies that

$$f'(x) = 0 \iff x = \frac{1}{C_d^4} = \int |\nabla W(x)|^2 dx.$$

On the other hand,

$$\begin{aligned} f'(x) &> 0, \text{ for } x < \frac{1}{C_d^4}, \\ f(0) &= 0, \quad f\left(\frac{1}{C_d^4}\right) = \frac{1}{4C_d^4} = E(W). \end{aligned}$$

Together with (3.3) and the fact that $a = \|\nabla u\|_{L^2}^2 \in [0, \frac{1}{C_d^4}]$, these imply that

$$\begin{aligned} \|\nabla u\|_{L^2}^2 = a &\leq (1 - \bar{\delta}) \frac{1}{C_d^4} = (1 - \bar{\delta}) \int |\nabla W|^2 dx, \quad \bar{\delta} = \delta_0^{1/2}, \\ E(u) &\geq f(a) \geq 0. \end{aligned}$$

Now define

$$g(x) = x - C_d^4 x^2.$$

From (3.2), we also have

$$\int |\nabla u|^2 dx - \iint \frac{|u(x)|^2 |u(y)|^2}{|x - y|^4} dxdy \geq \int |\nabla u|^2 dx - C_d^4 \left(\int |\nabla u|^2 dx \right)^2 = g(a). \quad (3.4)$$

Note that

$$\begin{aligned} g(x) &= 0 \iff x = 0, \text{ or } x = \frac{1}{C_d^4}, \\ g'(0) &= 1, \quad g'\left(\frac{1}{C_d^4}\right) = -1, \quad g''(x) = -2C_d^4 < 0. \end{aligned}$$

Hence, we obtain

$$g(x) \geq \frac{1}{2} \min \left(x, \frac{1}{C_d^4} - x \right) \quad \text{for } 0 \leq x \leq \frac{1}{C_d^4}.$$

Since $\|\nabla u\|_{L^2}^2 = a \in [0, (1 - \bar{\delta})\frac{1}{C_d^4}]$, the above inequality implies that

$$\begin{aligned} (\text{LHS}) \text{ of (3.4)} &\geq g(a) \geq \frac{1}{2} \min(a, \frac{1}{C_d^4} - a) \\ &\geq \frac{1}{2} \min(a, \bar{\delta}a) = \frac{\bar{\delta}}{2}a. \end{aligned}$$

This completes the proof.

Corollary 3.1. *Assume that $u \in \dot{H}^1(\mathbb{R}^d)$ and that $\|\nabla u\|_{L^2} < \|\nabla W\|_{L^2}$. Then $E(u) \geq 0$.*

Proof: If $E(u) < E(W)$, the conclusion follows from Lemma 3.1. If $E(u) \geq E(W) = \frac{1}{4C_d^4}$, it is clear.

Proposition 3.1 (Lower bound on the convexity of the variance). *Let u be a solution of (2.1) with $t_0 = 0, u(0) = u_0$ such that for $\delta_0 > 0$*

$$\int |\nabla u_0|^2 dx < \int |\nabla W|^2 dx, \quad E(u_0) < (1 - \delta_0)E(W).$$

Let $I \ni 0$ be the maximal interval of existence given by Definition 2.1. Let $\bar{\delta} = \delta_0^{1/2}$ be as in Lemma 3.1. Then for each $t \in I$, we have

$$\begin{aligned} \int |\nabla u(t)|^2 dx - \iint \frac{|u(t)|^2 |u(t)|^2}{|x-y|^4} dx dy &\geq \frac{\bar{\delta}}{2} \int |\nabla u(t)|^2 dx, \\ \int |\nabla u(t)|^2 dx &\leq (1 - \bar{\delta}) \int |\nabla W|^2 dx, \\ E(u(t)) &\geq 0. \end{aligned}$$

Proof: We prove it by the continuity argument. Define

$$\Omega = \{t \in I, \|\nabla u(t)\|_{L^2} < \|\nabla W\|_{L^2}, E(u(t)) < (1 - \delta_0)E(W)\}.$$

It suffices to prove that Ω is both open and closed.

Firstly, we see that $t_0 \in \Omega$. Secondly, Ω is open because of $u \in C_t^0(I, \dot{H}^1)$ and the conservation of energy. Lastly, we need to prove that Ω is also closed. For any $t_n \in \Omega, T \in I$, and $t_n \rightarrow T$. Then

$$\|\nabla u(t_n)\|_{L^2} < \|\nabla W\|_{L^2}, \quad E(u(t_n)) < (1 - \delta_0)E(W).$$

From Lemma 3.1, we obtain

$$\|\nabla u(t_n)\|_{L^2}^2 < (1 - \bar{\delta})\|\nabla W\|_{L^2}^2.$$

Using the fact that $u \in C_t^0(I, \dot{H}^1)$ and the conservation of energy again, we have

$$\|\nabla u(T)\|_{L^2}^2 \leq (1 - \bar{\delta})\|\nabla W\|_{L^2}^2, \quad E(u(T)) = E(u(t_n)) < (1 - \delta_0)E(W).$$

This implies that $T \in \Omega$ and completes the proof.

Corollary 3.2 (Comparability of gradient and energy). *Let u, u_0 be as in Proposition 3.1. Then for all $t \in I$ we have*

$$E(u(t)) \approx \int |\nabla u(t)|^2 dx \approx \int |\nabla u_0|^2 dx$$

with comparability constants which depend only on δ_0 .

Proof: From Proposition 3.1, we have

$$\begin{aligned} \frac{1}{2} \int |\nabla u(t)|^2 dx &\geq E(u(t)) = \frac{1}{4} \int |\nabla u(t)|^2 dx + \frac{1}{4} \left(\int |\nabla u(t, x)|^2 dx - \iint \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dxdy \right) \\ &\geq \frac{2 + \bar{\delta}}{8} \int |\nabla u(t)|^2 dx \quad \forall t \in I. \end{aligned}$$

This together with the conservation of energy implies the claim.

In order to obtain blow up results, we first give the (local) virial identity, which we can verify by some direct computations.

Lemma 3.2. *Let $\varphi \in C_0^\infty(\mathbb{R}^d)$, $V(x) = |x|^{-4}$, $t \in [0, T_+(u_0))$. Then*

$$\begin{aligned} (1) \quad \frac{d}{dt} \int |u|^2 \varphi dx &= 2 \operatorname{Im} \int \bar{u} \nabla u \nabla \varphi dx; \\ (2) \quad \frac{d^2}{dt^2} \int |u|^2 \varphi dx &= - \int \Delta \Delta \varphi |u|^2 dx + 4 \operatorname{Re} \int \varphi_{jk} \bar{u}_j u_k dx \\ &\quad - \operatorname{Re} \int \int (\nabla \varphi(x) - \nabla \varphi(y)) \nabla V(x - y) |u(y)|^2 |u(x)|^2 dxdy. \end{aligned}$$

Proposition 3.2. *Assume that $u_0 \in \dot{H}^1(\mathbb{R}^d)$ and*

$$E(u_0) < E(W), \quad \int |\nabla u_0|^2 dx > \int |\nabla W|^2 dx.$$

If $|x|u_0 \in L^2$ or u_0 is radial, then the maximal interval I of existence must be finite.

Proof: Indeed, we can choose a suitable small number $\delta_0 > 0$, such that

$$E(u_0) < (1 - \delta_0)E(W), \quad \int |\nabla u_0|^2 dx > \int |\nabla W|^2 dx.$$

Arguing as in Lemma 3.1, we obtain that there exists $\tilde{\delta}$ such that

$$\int |\nabla u_0|^2 dx > (1 + \tilde{\delta}) \int |\nabla W|^2 dx = \frac{1 + \tilde{\delta}}{C_d^4}.$$

This shows that

$$\begin{aligned} \int |\nabla u_0|^2 dx - \iint \frac{|u_0(x)|^2 |u_0(y)|^2}{|x - y|^4} dxdy &= 4E(u_0) - \int |\nabla u_0|^2 dx \\ &< 4(1 - \delta_0)E(W) - \frac{1 + \tilde{\delta}}{C_d^4} = \frac{1 - \delta_0}{C_d^4} - \frac{1 + \tilde{\delta}}{C_d^4} \\ &= -\frac{\delta_0 + \tilde{\delta}}{C_d^4} < 0. \end{aligned}$$

Now define

$$\Omega = \{t \in I, \|\nabla u(t)\|_{L^2} > \|\nabla W\|_{L^2}, E(u(t)) < (1 - \delta_0)E(W)\}.$$

Using the continuity argument and arguing as in Proposition 3.1, we have

$$\Omega = I.$$

Arguing as in Lemma 3.1 again, we have

$$\|\nabla u(t)\|_{L^2}^2 > (1 + \tilde{\delta})\|\nabla W\|_{L^2}^2.$$

Then

$$\int |\nabla u(t, x)|^2 dx - \iint \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dxdy = -\frac{\delta_0 + \tilde{\delta}}{C_d^4} < 0, \forall t \in I.$$

As for the case that $|x|u_0 \in L^2$. From Lemma 3.2, we have

$$\frac{d^2}{dt^2} \int |x|^2 |u(t, x)|^2 dx = 8 \left(\int |\nabla u(t, x)|^2 dx - \iint \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dxdy \right) < 0.$$

This implies that I must be finite.

As for the case that u_0 is radial. Using the local virial identity [2], [3] and [29], we can also deduce the same result.

4 Existence and compactness of a critical element

Let us consider the statement

(SC) For all $u_0 \in \dot{H}^1(\mathbb{R}^d)$ with $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_0) < E(W)$, if u is the corresponding solution to (2.1), with maximal interval of existence I , then $I = (-\infty, +\infty)$ and $\|u\|_{X(\mathbb{R})} < +\infty$.

We say that $(SC)(u_0)$ holds if for this particular u_0 with $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_0) < E(W)$, and u is the corresponding solution to (2.1), with maximal interval of existence I , then $I = (-\infty, +\infty)$ and $\|u\|_{X(\mathbb{R})} < +\infty$.

Note that, because of Remark 2.1, if $\|u_0\|_{\dot{H}^1} \leq \tilde{\delta}$, $(SC)(u_0)$ holds. Thus, in light of Corollary 3.2, there exists $\eta_0 > 0$ such that if u_0 is as in (SC) and $E(u_0) < \eta_0$, then $(SC)(u_0)$ holds. Moreover, $E(u_0) \geq 0$ in light of Proposition 3.1. Thus, there exists a number E_c , with $\eta_0 \leq E_c \leq E(W)$, such that, if u_0 is radial with $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_0) < E_c$, then $(SC)(u_0)$ holds, and E_c is optimal with this property. If $E_c \geq E(W)$, then the first part of Theorem 1.1 is true. For the rest of this section, we will assume that $E_c < E(W)$ and ultimately deduce a contradiction in Section 5. By definition of E_c , we have

(C.1) If u_0 is radial and $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_0) < E_c$, then $(SC)(u_0)$ holds.

(C.2) There exists a sequence of radial solutions u_n to (2.1) with corresponding initial data $u_{n,0}$ such that $\|\nabla u_{n,0}\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_{n,0}) \searrow E_c$ as $n \rightarrow +\infty$, for which $(SC)(u_{n,0})$ does not hold for any n .

The goal of this section is to use the above sequence $u_{n,0}$ to prove the existence of an \dot{H}^1 radial solution u_c to (2.1) with initial data $u_{c,0}$ such that $\|\nabla u_{c,0}\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_{c,0}) = E_c$ for which $(SC)(u_{c,0})$ does not hold (see Proposition 4.1). Moreover, we will show that this critical solution has a compactness property up to the symmetries of this equation (see Proposition 4.2).

Before stating and proving Proposition 4.1, we introduce some useful preliminaries in the spirit of the results of Keraani [14]. First we give the profile decomposition lemma.

Lemma 4.1 (Profile decomposition). *Let $v_{n,0}$ be a radial uniformly bounded sequence in \dot{H}^1 , i.e. $\|\nabla v_{n,0}\|_{L^2} \leq A$. Assume that $\|e^{it\Delta}v_{n,0}\|_{X(\mathbb{R})} \geq \delta > 0$, where $\delta = \delta(d)$ is as in Proposition 2.1. Then for each J , there exists a subsequence of $v_{n,0}$, also denoted $v_{n,0}$, and*

(1) *For each $1 \leq j \leq J$, there exists a radial profile $V_{0,j}$ in \dot{H}^1 .*

(2) *For each $1 \leq j \leq J$, there exists a sequence of $(\lambda_{j,n}, t_{j,n})$ with*

$$\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{j',n}|}{\lambda_{j,n}^2} \rightarrow \infty \quad \text{as } n \rightarrow \infty \quad \text{for } j \neq j'. \quad (4.1)$$

(3) *There exists a sequence of radial remainder w_n^J in \dot{H}^1 ,*

such that

$$\begin{aligned} v_{n,0}(x) &= \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{(d-2)/2}} e^{-it_{j,n}\Delta} V_{0,j}\left(\frac{x}{\lambda_{j,n}}\right) + w_n^J(x) \\ &= \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\right) + w_n^J(x) \end{aligned} \quad (4.2)$$

with

$$V_j^l(t, x) = e^{it\Delta} V_{0,j}(x), \quad \|V_{0,1}\|_{\dot{H}^1} \geq \alpha_0(A) > 0, \quad (4.3)$$

$$\|\nabla v_{n,0}\|_{L^2}^2 = \sum_{j=1}^J \|\nabla V_{0,j}\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 + o_n(1), \quad (4.4)$$

$$E(v_{n,0}) = \sum_{j=1}^J E(V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}^2}\right)) + E(w_n^J) + o_n(1), \quad (4.5)$$

$$\lim_{J \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \|e^{it\Delta} w_n^J\|_{L^q(\mathbb{R}, L^r)} \right] = 0, \quad \forall \frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right) - 1, \quad \frac{2d}{d-2} \leq r < \frac{2d}{d-4}. \quad (4.6)$$

Proof: Here we only give the proof of energy asymptotic Pythagorean expansion (4.5), the rest is standard (see [14]).

By the asymptotic Pythagorean expansion of kinetic energy, it suffices to show that

$$\begin{aligned} \iint \frac{1}{|x-y|^4} |v_{n,0}(x)|^2 |v_{n,0}(y)|^2 \, dx dy &= \sum_{j=1}^J \iint \frac{1}{|x-y|^4} |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2 \, dx dy \\ &\quad + \iint \frac{1}{|x-y|^4} |w_n^J(x)|^2 |w_n^J(y)|^2 \, dx dy + o_n(1), \quad \forall J \geq 1. \end{aligned}$$

We first claim that if $J \geq 1$ is fixed, the orthogonality condition (4.1) implies that

$$\begin{aligned} &\iint \frac{1}{|x-y|^4} \left| \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}) \right|^2 \left| \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{y}{\lambda_{j,n}}) \right|^2 \, dx dy \\ &= \sum_{j=1}^J \iint \frac{1}{|x-y|^4} |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2 \, dx dy + o_n(1). \end{aligned} \tag{4.7}$$

By reindexing, we can arrange such that there is $J_0 \leq J$ with

- (1) $\forall 1 \leq j \leq J_0$, we have that $\left| \frac{t_{j,n}}{\lambda_{j,n}^2} \right| \leq C$ in n ;
- (2) $\forall J_0 + 1 \leq j \leq J$, we have that $\left| \frac{t_{j,n}}{\lambda_{j,n}^2} \right| \rightarrow +\infty$ as $n \rightarrow +\infty$.

By passing to a subsequence and adjusting the profile $V_{0,j}$, we may assume that

$$\forall 1 \leq j \leq J_0, \quad \frac{t_{j,n}}{\lambda_{j,n}^2} = 0,$$

From case (2), we have

$$\lim_{n \rightarrow +\infty} \iint \frac{1}{|x-y|^4} |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2 \, dx dy = 0, \quad \forall J_0 + 1 \leq j \leq J. \tag{4.8}$$

Indeed, using Hardy inequality and the decay estimates for the free Schrödinger equation (similar to Lemma 4.1 in [5] and Corollary 2.3.7 in [1]), we have for $J_0 + 1 \leq j \leq J$

$$\iint \frac{1}{|x-y|^4} |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2 \, dx dy \lesssim \|V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2})\|_{L^{\frac{2d}{d-2}}}^4 \rightarrow 0, \quad n \rightarrow +\infty.$$

By (4.1), if $1 \leq j < k \leq J_0$, we have

$$\frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{\lambda_{k,n}}{\lambda_{j,n}} \rightarrow \infty \quad \text{as } n \rightarrow \infty. \tag{4.9}$$

This implies that

$$\begin{aligned} & \iint \frac{1}{|x-y|^4} \left| \sum_{j=1}^{J_0} \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_{0,j} \left(\frac{x}{\lambda_{j,n}} \right) \right|^2 \left| \sum_{j=1}^{J_0} \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_{0,j} \left(\frac{y}{\lambda_{j,n}} \right) \right|^2 dx dy \\ &= \sum_{j=1}^{J_0} \iint \frac{1}{|x-y|^4} |V_{0,j}(x)|^2 |V_{0,j}(y)|^2 dx dy + o_n(1). \end{aligned} \quad (4.10)$$

Hence, from (4.8) and (4.10), we obtain

$$\begin{aligned} & \iint \frac{1}{|x-y|^4} \left| \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}} \right) \right|^2 \left| \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{y}{\lambda_{j,n}} \right) \right|^2 dx dy \\ &= \iint \frac{1}{|x-y|^4} \left| \sum_{j=1}^{J_0} \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}} \right) + \sum_{j=J_0+1}^J \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}} \right) \right|^2 \\ &\quad \times \left| \sum_{j=1}^{J_0} \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{y}{\lambda_{j,n}} \right) + \sum_{j=J_0+1}^J \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{y}{\lambda_{j,n}} \right) \right|^2 dx dy \\ &= \iint \frac{1}{|x-y|^4} \left| \sum_{j=1}^{J_0} \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_{0,j} \left(\frac{x}{\lambda_{j,n}} \right) + \sum_{j=J_0+1}^J \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}} \right) \right|^2 \\ &\quad \times \left| \sum_{j=1}^{J_0} \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_{0,j} \left(\frac{y}{\lambda_{j,n}} \right) + \sum_{j=J_0+1}^J \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{y}{\lambda_{j,n}} \right) \right|^2 dx dy \\ &= \iint \frac{1}{|x-y|^4} \left| \sum_{j=1}^{J_0} \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_{0,j} \left(\frac{x}{\lambda_{j,n}} \right) \right|^2 \left| \sum_{j=1}^{J_0} \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_{0,j} \left(\frac{y}{\lambda_{j,n}} \right) \right|^2 dx dy \\ &\quad + \sum_{j=J_0+1}^J \iint \frac{1}{|x-y|^4} |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2 dx dy + o_n(1) \\ &= \sum_{j=1}^J \iint \frac{1}{|x-y|^4} |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2 dx dy + o_n(1), \end{aligned}$$

this yields (4.7).

Secondly, we claim that

$$\lim_{n \rightarrow +\infty} \|w_n^J(x)\|_{L_x^{\frac{2d}{d-2}}} = 0 \quad \text{as } J \rightarrow +\infty. \quad (4.11)$$

Indeed, we have

$$\|w_n^J(x)\|_{L_x^{\frac{2d}{d-2}}} \lesssim \|e^{it\Delta} w_n^J(x)\|_{L_t^\infty(\mathbb{R}; L_x^{\frac{2d}{d-2}})},$$

this together with (4.6) implies the claim.

Note that (4.11) implies that $\{w_n^J\}$ is uniformly bounded in $L^{\frac{2d}{d-2}}(\mathbb{R}^d)$, the uniform boundness of $\{v_{n,0}\}$ in $\dot{H}^1(\mathbb{R}^d)$ also implies uniformly bounded in $L^{\frac{2d}{d-2}}(\mathbb{R}^d)$. Thus we can choose $J_1 \geq J$ and N_1 such that for $n \geq N_1$, we have

$$\begin{aligned} & \left| \iint \frac{|v_{n,0}(x)|^2 |v_{n,0}(y)|^2}{|x-y|^4} dx dy - \iint \frac{|v_{n,0}(x) - w_n^{J_1}(x)|^2 |v_{n,0}(y) - w_n^{J_1}(y)|^2}{|x-y|^4} dx dy \right| \\ & + \left| \iint \frac{|w_n^J(x) - w_n^{J_1}(x)|^2 |w_n^J(y) - w_n^{J_1}(y)|^2}{|x-y|^4} dx dy - \iint \frac{|w_n^J(x)|^2 |w_n^J(y)|^2}{|x-y|^4} dx dy \right| \\ & \leq C \left(\sup_n \|v_{n,0}(x)\|_{L^{\frac{2d}{d-2}}}^3 + \sup_n \|w_n^J(x)\|_{L^{\frac{2d}{d-2}}}^3 \right) \|w_n^{J_1}(x)\|_{L^{\frac{2d}{d-2}}} + C \|w_n^{J_1}(x)\|_{L^{\frac{2d}{d-2}}}^4 \leq \epsilon. \end{aligned} \quad (4.12)$$

By (4.7), we get $N_2 \geq N_1$ such that for $n \geq N_2$

$$\begin{aligned} & \left| \iint \frac{|v_{n,0}(x) - w_n^{J_1}(x)|^2 |v_{n,0}(y) - w_n^{J_1}(y)|^2}{|x-y|^4} dx dy \right. \\ & \quad \left. - \sum_{j=1}^{J_1} \iint \frac{|V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2}{|x-y|^4} dx dy \right| \leq \epsilon. \end{aligned} \quad (4.13)$$

Using (4.2), we have

$$w_n^J(x) - w_n^{J_1}(x) = \sum_{j=J+1}^{J_1} \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\right).$$

By (4.7), we get $N_3 \geq N_2$ such that for $n \geq N_3$

$$\left| \iint \frac{|w_n^J(x) - w_n^{J_1}(x)|^2 |w_n^J(y) - w_n^{J_1}(y)|^2}{|x-y|^4} dx dy - \sum_{j=J+1}^{J_1} \iint \frac{|V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2}{|x-y|^4} dx dy \right| \leq \epsilon.$$

Combining the above inequality with (4.12), (4.13), we obtain that for $n \geq N_3$

$$\begin{aligned} & \left| \iint \frac{|v_{n,0}(x)|^2 |v_{n,0}(y)|^2}{|x-y|^4} dx dy - \sum_{j=1}^J \iint \frac{|V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2}{|x-y|^4} dx dy \right. \\ & \quad \left. - \iint \frac{|w_n^J(x)|^2 |w_n^J(y)|^2}{|x-y|^4} dx dy \right| \\ & = \left| \iint \frac{|v_{n,0}(x)|^2 |v_{n,0}(y)|^2}{|x-y|^4} dx dy - \iint \frac{|v_{n,0}(x) - w_n^{J_1}(x)|^2 |v_{n,0}(y) - w_n^{J_1}(y)|^2}{|x-y|^4} dx dy \right. \\ & \quad + \iint \frac{|v_{n,0}(x) - w_n^{J_1}(x)|^2 |v_{n,0}(y) - w_n^{J_1}(y)|^2}{|x-y|^4} dx dy - \sum_{j=1}^{J_1} \iint \frac{|V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2}{|x-y|^4} dx dy \\ & \quad + \iint \frac{|w_n^J(x) - w_n^{J_1}(x)|^2 |w_n^J(y) - w_n^{J_1}(y)|^2}{|x-y|^4} dx dy - \iint \frac{|w_n^J(x)|^2 |w_n^J(y)|^2}{|x-y|^4} dx dy \\ & \quad + \sum_{j=J+1}^{J_1} \iint \frac{|V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x)|^2 |V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y)|^2}{|x-y|^4} dx dy - \iint \frac{|w_n^J(x) - w_n^{J_1}(x)|^2 |w_n^J(y) - w_n^{J_1}(y)|^2}{|x-y|^4} dx dy \\ & \leq 3\epsilon, \end{aligned}$$

this completes the proof.

Lemma 4.2. *Let $\{z_{0,n}\} \in \dot{H}^1$ be radial, with*

$$\|\nabla z_{0,n}\|_{L^2} < \|\nabla W\|_{L^2}, \quad E(z_{0,n}) \rightarrow E_c.$$

and with $\|e^{it\Delta} z_{0,n}\|_{X(\mathbb{R})} \geq \delta > 0$, where $\delta = \delta(\|\nabla W\|_{L^2})$ is as in Proposition 2.1. Let $V_{0,j}$ be as in Lemma 4.1. Assume that one of the two hypotheses holds

$$(1) \quad \lim_{n \rightarrow +\infty} E(V_1^l(-\frac{t_{1,n}}{\lambda_{1,n}^2})) < E_c. \quad (4.14)$$

(2) *After passing to a subsequence, we have that*

$$\lim_{n \rightarrow +\infty} E(V_1^l(-\frac{t_{1,n}}{\lambda_{1,n}^2})) = E_c \quad (4.15)$$

with $s_{1,n} = -\frac{t_{1,n}}{\lambda_{1,n}^2} \rightarrow s_ \in [-\infty, \infty]$, and if U_1 is the nonlinear profile associated to $(V_{0,1}, \{s_{1,n}\})$, we have that the maximal interval of existence of U_1 is $I = (-\infty, +\infty)$ and $\|U_1\|_{X(\mathbb{R})} < \infty$.*

Then, after passing to a subsequence, for n large, if z_n is the solution of (2.1) with data at $t = 0$ equal to $z_{0,n}$, then $(SC)(z_{0,n})$ holds.

Proof: Case 2 holds. Applying Lemma 4.1 to $\{z_{0,n}\}$, we have

$$\begin{aligned} z_{0,n}(x) &= \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\right) + w_n^J \\ \|\nabla W\|_{L^2}^2 > \|\nabla z_{0,n}\|_{L^2}^2 &= \sum_{j=1}^J \|\nabla V_{0,j}\|_{L^2}^2 + \|w_n^J\|_{L^2}^2 + o_n(1), \\ &= \sum_{j=1}^J \|\nabla V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2})\|_{L^2}^2 + \|w_n^J\|_{L^2}^2 + o_n(1), \end{aligned} \quad (4.16)$$

$$E_c \leftarrow E(z_{0,n}) = \sum_{j=1}^J E(V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2})) + E(w_n^J) + o_n(1). \quad (4.17)$$

By (4.16) and Corollary 3.1, we have for every $1 \leq j \leq J$

$$E(V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2})) \geq 0, \quad E(w_n^J) \geq 0.$$

Using (4.15) and (4.17), we have for every $2 \leq j \leq J$

$$E(V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2})) \rightarrow 0, \quad E(w_n^J) \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Using Corollary 3.2, we obtain that

$$\sum_{j=2}^J \|\nabla V_{0,j}\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 = \sum_{j=2}^J \|\nabla V_j^l((-\frac{t_{j,n}}{\lambda_{j,n}^2}))\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Hence, we have for every $2 \leq j \leq J$

$$V_{0,j} \equiv 0, \quad \text{and} \quad \|\nabla w_n^J\|_{L^2} \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Therefore,

$$z_{0,n}(x) = \frac{1}{\lambda_{1,n}^{(d-2)/2}} V_1^l(s_{1,n}, \frac{x}{\lambda_{1,n}}) + w_n, \text{ where } \|\nabla w_n\|_{L^2} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Let $v_{0,n} = \lambda_{1,n}^{(d-2)/2} z_{0,n}(\lambda_{1,n}x)$, $\tilde{w}_n = \lambda_{1,n}^{(d-2)/2} w_n(\lambda_{1,n}x)$, we have $\|\nabla v_{0,n}\|_{L^2} = \|\nabla z_{0,n}\|_{L^2} < \|\nabla W\|_{L^2}$ and

$$v_{0,n}(x) = V_1^l(s_{1,n}, x) + \tilde{w}_n(x), \text{ where } \|\nabla \tilde{w}_n\|_{L^2} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Note that by the definition of nonlinear profile, we have

$$\|\nabla U_1(s_{1,n}) - \nabla V_1^l(s_{1,n})\|_{L^2} \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

then

$$\begin{aligned} v_{0,n}(x) &= U_1(s_{1,n}, x) + \tilde{w}_n, \quad \|\nabla \tilde{w}_n\|_{L^2} \rightarrow 0. \\ E(U_1(s_{1,n})) &= E(V_1^l(s_{1,n})) + o_n(1) \rightarrow E_c, \\ \|\nabla U_1(s_{1,n})\|_{L^2} &= \|\nabla V_1^l(s_{1,n})\|_{L^2} + o_n(1) \\ &= \|\nabla V_{0,1}\|_{L^2} + o_n(1) < \|\nabla W\|_{L^2}. \end{aligned}$$

We now apply Proposition 2.3 with $\tilde{u} = U_1, e = 0$ to obtain that $(SC)(v_{0,n})$ holds, then this case follows from the dilation invariance of (2.1).

Case 1 holds. We first claim that

$$\lim_{n \rightarrow +\infty} E(V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2})) < E_c \quad \text{for } j \geq 2. \quad (4.18)$$

After passing to a subsequence, we assume that

$$\lim_{n \rightarrow \infty} E(V_1^l(-\frac{t_{1,n}}{\lambda_{1,n}^2})) < E_c. \quad (4.19)$$

Applying Lemma 4.1 to $\{z_{0,n}\}$, we have

$$\begin{aligned} z_{0,n}(x) &= \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}} \right) + w_n^J \\ \|\nabla W\|_{L^2}^2 > \|\nabla z_{0,n}\|_{L^2}^2 &= \sum_{j=1}^J \|\nabla V_{0,j}\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 + o_n(1), \\ &= \sum_{j=1}^J \|\nabla V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 + o_n(1), \end{aligned} \quad (4.20)$$

$$E_c \leftarrow E(z_{0,n}) = \sum_{j=1}^J E(V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)) + E(w_n^J) + o_n(1). \quad (4.21)$$

By (4.20) and Corollary 3.1, we have for every $1 \leq j \leq J$

$$E(V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)) \geq 0, \quad E(w_n^J) \geq 0.$$

Note that

$$\begin{aligned} \|\nabla V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)\|_{L^2}^2 &= \|\nabla V_{0,1}\|_{L^2}^2 < \|\nabla W\|_{L^2}^2, \\ E(V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)) &\leq E_c + o_n(1) < E(W). \end{aligned}$$

Hence, from Lemma 3.1 and Lemma 4.1, we have

$$\begin{aligned} &\int |\nabla V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)|^2 dx - \int \int \frac{|V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x \right)|^2 |V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y \right)|^2}{|x-y|^4} dxdy \geq \frac{\bar{\delta}}{2} \int |\nabla V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)|^2 dx, \\ E(V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)) &= \frac{1}{4} \|\nabla V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)\|_{L^2}^2 + \frac{1}{4} \left(\int |\nabla V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)|^2 dx \right. \\ &\quad \left. - \int \int \frac{|V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, x \right)|^2 |V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, y \right)|^2}{|x-y|^4} dxdy \right) \\ &\geq C \|\nabla V_1^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)\|_{L^2}^2 = C \|\nabla V_{0,1}\|_{L^2}^2 \geq C\alpha_0 > 0. \end{aligned}$$

By (4.21), we have

$$E_c \leftarrow E(z_{0,n}) \geq C\alpha_0 + \sum_{j=2}^J E(V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)) + E(w_n^J) + o_n(1),$$

which implies the claim.

After passing to a subsequence, we can assume that for any $j \geq 1$

$$\lim_{n \rightarrow \infty} E(V_j^l((-\frac{t_{j,n}}{\lambda_{j,n}^2}))) \text{ exists, and } \lim_{n \rightarrow \infty} -\frac{t_{j,n}}{\lambda_{j,n}^2} = \bar{s}_j \in [-\infty, \infty].$$

If U_j is the nonlinear profile associated to $(V_{0,j}, \{-\frac{t_{j,n}}{\lambda_{j,n}^2}\})$, then by the definition of nonlinear profile, for sufficiently large n , we obtain

$$\begin{aligned} \|\nabla U_j(-\frac{t_{j,n}}{\lambda_{j,n}^2})\|_{L^2}^2 &= \|\nabla V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2})\|_{L^2}^2 + o_n(1) < \|\nabla W\|_{L^2}^2, \\ E(U_j(-\frac{t_{j,n}}{\lambda_{j,n}^2})) &= E(V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2})) + o_n(1) < E_c. \end{aligned}$$

By the definition of E_c , we have that U_j satisfies (SC) . Moreover we also have $\|U_j\|_{W(\mathbb{R})} < \infty$, and we obtain from Corollary 3.2

$$E(U_j(t)) \approx \|\nabla U_j(t)\|_{L^2}^2 \approx \|\nabla U_j(0)\|_{L^2}^2, \quad \forall t \in \mathbb{R}. \quad (4.22)$$

On the other hand, we claim that there exists j_0 such that, for $j \geq j_0$

$$\|U_j\|_{X(\mathbb{R})} \leq C \|\nabla V_{0,j}\|_{L^2}. \quad (4.23)$$

In fact, from (4.20), we have

$$\sum_{j=1}^J \|\nabla V_{0,j}\|_{L^2}^2 \leq \|\nabla z_{0,n}\|_{L^2}^2 + o_n(1) \leq \|\nabla W\|_{L^2}^2,$$

then there exists j_0 , for $j \geq j_0$, such that $\|\nabla V_{0,j}\|_{L^2} \leq \tilde{\delta}$, where $\tilde{\delta}$ is so small that $\|e^{it\Delta} V_{0,j}\|_{X(\mathbb{R})} \leq \delta$, with δ as in Proposition 2.1. Note that

$$U_j(t) = e^{it\Delta} V_{0,j} + i \int_{\bar{s}_j}^t e^{i(t-s)\Delta} (|x|^{-4} * |U_j|^2)(s, x) U_j(s, x) ds,$$

this together with the local wellposedness theory implies

$$\|U_j\|_{X(\mathbb{R})} \leq C \|\nabla V_{0,j}\|_{L^2}.$$

Since for sufficiently large n , we have

$$z_{0,n}(x) = \sum_{j=1}^{J(\epsilon_0)} \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}} \right) + w_n^{J(\epsilon_0)}, \quad \|e^{it\Delta} w_n^{J(\epsilon_0)}\|_{Z^1(\mathbb{R})} \leq \epsilon_0.$$

Define the near-solution

$$H_{n,\epsilon_0}(t, x) = \sum_{j=1}^{J(\epsilon_0)} \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} U_j \left(\frac{t - t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}} \right).$$

Then H_{n,ϵ_0} satisfies the following equation

$$\begin{aligned}(i\partial_t + \Delta)H_{n,\epsilon_0}(t, x) &= \sum_{j=1}^{J(\epsilon_0)} f\left(\frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} U_j\left(\frac{t-t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\right)\right) \\ &= f(H_{n,\epsilon_0}(t, x)) + R_{n,\epsilon_0}(t, x)\end{aligned}$$

where

$$R_{n,\epsilon_0}(t, x) = \sum_{j=1}^{J(\epsilon_0)} f\left(\frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} U_j\left(\frac{t-t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\right)\right) - f(H_{n,\epsilon_0}(t, x)).$$

By the definition of the nonlinear profile U_j , we have

$$\begin{aligned}z_{0,n}(x) &= \sum_{j=1}^{J(\epsilon_0)} \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} U_j\left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\right) + \tilde{w}_n^{J(\epsilon_0)} \\ &= H_{n,\epsilon_0}(0) + \tilde{w}_n^{J(\epsilon_0)}, \quad \|e^{it\Delta} \tilde{w}_n^{J(\epsilon_0)}\|_{\mathcal{Z}^1(\mathbb{R})} \leq 2\epsilon_0 \text{ for } n \gg 1.\end{aligned}$$

By the orthogonality property and (4.20), we have

$$\|\nabla H_{n,\epsilon_0}(0)\|_{L^2}^2 \leq C \sum_{j=1}^{J(\epsilon_0)} \|\nabla V_j^l\left(-\frac{t_{j,n}}{\lambda_{j,n}^2}\right)\|_{L^2}^2 + o_n(1) \leq C \|\nabla W\|_{L^2}^2. \quad (4.24)$$

In addition, we also have

$$\begin{aligned}\|H_{n,\epsilon_0}\|_{X(\mathbb{R})}^6 &= \int \left\| \sum_{j=1}^{J(\epsilon_0)} \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} U_j\left(\frac{t-t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\right) \right\|_{L_x^{\frac{6d}{3d-8}}}^6 dt \\ &\leq \int \left(\sum_{j=1}^{J(\epsilon_0)} \left\| \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} U_j\left(\frac{t-t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\right) \right\|_{L_x^{\frac{6d}{3d-8}}} \right)^6 dt \\ &\leq \sum_{j=1}^{J(\epsilon_0)} \int \left\| \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} U_j\left(\frac{t-t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\right) \right\|_{L_x^{\frac{6d}{3d-8}}}^6 dt \\ &\quad + C_{J(\epsilon_0)} \sum_{j \neq j'} \int \left\| \frac{1}{\lambda_{j,n}^{\frac{d-2}{2}}} U_j\left(\frac{t-t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}}\right) \right\|_{L_x^{\frac{6d}{3d-8}}} \left\| \frac{1}{\lambda_{j',n}^{\frac{d-2}{2}}} U_{j'}\left(\frac{t-t_{j',n}}{\lambda_{j',n}^2}, \frac{x}{\lambda_{j',n}}\right) \right\|_{L_x^{\frac{6d}{3d-8}}}^5 dt \\ &= I + II.\end{aligned}$$

For the first term, from (4.20) and (4.23), we have

$$\begin{aligned}I &\leq \sum_{j=1}^{j_0} \|U_j\|_{X(\mathbb{R})}^6 + \sum_{j=j_0+1}^{J(\epsilon_0)} \|U_j\|_{X(\mathbb{R})}^6 \\ &\leq \sum_{j=1}^{j_0} \|U_j\|_{X(\mathbb{R})}^6 + C \sum_{j=j_0+1}^{J(\epsilon_0)} \|\nabla V_{0,j}\|_{L^2}^6 \leq \sum_{j=1}^{j_0} \|U_j\|_{X(\mathbb{R})}^6 + C \left(\sum_{j=j_0+1}^{J(\epsilon_0)} \|\nabla V_{0,j}\|_{L^2}^2 \right)^3 \\ &\leq \frac{C_0}{2},\end{aligned}$$

where C_0 is independent of $J(\epsilon_0)$. For the second term, we have from the orthogonality of $(\lambda_{j,n}, t_{j,n})$

$$II \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, we obtain

$$\|H_{n,\epsilon_0}\|_{X(\mathbb{R})}^6 \leq C_0, \quad \text{for } n \text{ sufficiently large,} \quad (4.25)$$

where C_0 is independent of $J(\epsilon_0)$.

Note that $\|U_j\|_{X(\mathbb{R})} < \infty$ and $\|U_j\|_{W(\mathbb{R})} < \infty$, using the orthogonality of $(\lambda_{j,n}, t_{j,n})$ again, we have that

$$\|R_{n,\epsilon_0}(t, x)\|_{L^{\frac{3}{2}}(\dot{H}^1, \frac{6d}{3d+4})} \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.26)$$

Last, for sufficiently large n , we have

$$\|\nabla z_{0,n} - \nabla H_{n,\epsilon_0}(0)\|_{L^2} \leq \|\tilde{w}_n^{J(\epsilon_0)}\|_{L^2} \leq \|w_n^{J(\epsilon_0)}\|_{L^2} + o_n(1) \leq \|\nabla W\|_{L^2}^2. \quad (4.27)$$

Combining Proposition 2.3, Remark 2.5 with (4.24)-(4.27), we obtain that $(SC)(z_{0,n})$ holds.

Proposition 4.1 (Existence of a critical solution). *There exists a radial solution u_c of (2.1) in \dot{H}^1 with data $u_{c,0}$ and maximal interval of existence I such that*

$$\|\nabla u_{c,0}\|_{L^2} < \|\nabla W\|_{L^2}, \quad E(u_{c,0}) = E_c$$

and

$$\|u_c\|_{X(I)} = +\infty.$$

Proof: By the definition of E_c and the assumption that $E_c < E(W)$, we can find $u_{0,n} \in \dot{H}^1$ radial, with $\|\nabla u_{0,n}\|_{L^2} < \|\nabla W\|_{L^2}$, $E(u_{0,n}) \searrow E_c$, and such that if u_n is the solution of (2.1) with data $u_{0,n}$ at $t = 0$ and maximal interval of existence $I_n = (-T_-(u_{0,n}), T_+(u_{0,n}))$, then

$$\|e^{it\Delta} u_{0,n}\|_{X(\mathbb{R})} \geq \delta \quad \text{as Proposition 2.1, and } \|u_n\|_{S(I_n)} = +\infty.$$

Note that $E_c < E(W)$, then there exists $\delta_0 > 0$, so that for sufficiently large n , we have $E(u_{0,n}) < (1 - \delta_0)E(W)$. By Proposition 3.1, we can find $\bar{\delta}$ so that

$$\|\nabla u_n(t)\|_{L^2}^2 \leq (1 - \bar{\delta})\|\nabla W\|_{L^2}^2, \quad \forall t \in I_n.$$

Applying Lemma 4.1 to $\{u_{0,n}\}$, we have

$$\begin{aligned} u_{0,n} &= \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}} \right) + w_n^J \\ \|V_{0,1}\|_{\dot{H}^1} &\geq \alpha_0(A) > 0, \quad \lim_{t \rightarrow \infty} \left[\lim_{n \rightarrow \infty} \|e^{it\Delta} w_n^J\|_{X(\mathbb{R})} \right] = 0, \end{aligned}$$

$$\begin{aligned}
(1 - \bar{\delta}) \|\nabla W\|_{L^2}^2 &\geq \|\nabla u_{0,n}\|_{L^2}^2 = \sum_{j=1}^J \|\nabla V_{0,j}\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 + o_n(1), \\
&= \sum_{j=1}^J \left\| \nabla V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right) \right\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 + o_n(1), \quad (4.28) \\
E_c \swarrow E(u_{0,n}) &= \sum_{j=1}^J E(V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)) + E(w_n^J) + o_n(1). \quad (4.29)
\end{aligned}$$

Because of (4.28), we have that

$$\left\| \nabla V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right) \right\|_{L^2}^2 \leq (1 - \frac{\bar{\delta}}{2}) \|\nabla W\|_{L^2}^2, \quad \|\nabla w_n^J\|_{L^2}^2 \leq (1 - \frac{\bar{\delta}}{2}) \|\nabla W\|_{L^2}^2, \quad \text{for } n \gg 1.$$

From Corollary 3.1, it follows that

$$E(V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2} \right)) \geq 0, \quad E(w_n^J) \geq 0.$$

By (4.29), we have that

$$E(V_1^l \left(-\frac{t_{1,n}}{\lambda_{1,n}^2} \right)) \leq E(u_{0,n}) + o_n(1),$$

therefore,

$$\liminf_{n \rightarrow \infty} E(V_1^l \left(-\frac{t_{1,n}}{\lambda_{1,n}^2} \right)) \leq E_c.$$

Note that $(SC)(u_{0,n})$ does not hold, we have from Lemma 4.2

$$\liminf_{n \rightarrow \infty} E(V_1^l \left(-\frac{t_{1,n}}{\lambda_{1,n}^2} \right)) = E_c.$$

Arguing as in the proof of Case 2, Lemma 4.2, we see that $\liminf_{n \rightarrow \infty} E(V_1^l \left(-\frac{t_{1,n}}{\lambda_{1,n}^2} \right)) = E_c$ and $E_c < E(W)$ imply that $J = 1$ and $\|\nabla w_n^J\|_{L^2} \rightarrow 0$ as $n \rightarrow +\infty$.

Thus

$$u_{0,n}(x) = \frac{1}{\lambda_{1,n}^{(d-2)/2}} V_1^l \left(-\frac{t_{1,n}}{\lambda_{1,n}^2}, \frac{x}{\lambda_{1,n}} \right) + w_n(x), \quad \|\nabla w_n\|_{L^2} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Let

$$v_{0,n}(x) = \lambda_{1,n}^{(d-2)/2} u_{0,n}(\lambda_{1,n} x), \quad \tilde{w}_n(x) = \lambda_{1,n}^{(d-2)/2} w_n(\lambda_{1,n} x),$$

then

$$v_{0,n}(x) = V_1^l \left(-\frac{t_{1,n}}{\lambda_{1,n}^2}, x \right) + \tilde{w}_n(x), \quad \|\nabla \tilde{w}_n\|_{L^2} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Let U_1 be the nonlinear profile associated to $(V_{0,1}, -\frac{t_{1,n}}{\lambda_{1,n}^2})$ and let I_1 be its maximal interval of existence. By the definition of the nonlinear profile, we have for $s_n = -\frac{t_{1,n}}{\lambda_{1,n}^2}$

$$\begin{aligned}\|\nabla U_1(s_n)\|_{L^2}^2 &= \|\nabla V_1^l(s_n)\|_{L^2}^2 + o_n(1) < \|\nabla W\|_{L^2}^2, \\ E(U_1(s_n)) &= E(V_1^l(s_n)) + o_n(1) = E_c + o_n(1).\end{aligned}$$

Let's fix $s_* \in I_1$. then from the conservation of energy and Proposition 3.1, we have

$$\|\nabla U_1(s_*)\|_{L^2}^2 < \|\nabla W\|_{L^2}^2, \quad E(U_1(s_*)) = E_c.$$

If $\|U_1\|_{X(I_1)} < +\infty$, Proposition 2.2 implies that $I_1 = (-\infty, +\infty)$, then $(SC)(u_{0,n})$ holds from Lemma 4.2, this obtains a contradiction. Thus

$$\|U_1\|_{X(I_1)} = +\infty.$$

This completes the proof.

Proposition 4.2 (Pre-compactness of the flow of the critical solution). *Let u_c be as in Proposition 4.1, and that $\|u_c\|_{X(I_+)} = +\infty$, where $I_+ = (0, +\infty) \cap I$. Then for $t \in I_+$, there exists $\lambda(t) \in \mathbb{R}^+$, such that K is precompact in \dot{H}^1 where*

$$K = \left\{ v(t, x), v(t, x) = \frac{1}{\lambda(t)^{\frac{d-2}{2}}} u_c(t, \frac{x}{\lambda(t)}), t \in I_+ \right\}.$$

Proof: For brevity of notation, let us set $u(t, x) = u_c(t, x)$. We argue by contradiction. If not, there exist $\eta_0 > 0$ and a sequence $\{t_n\}_{n=1}^\infty, t_n \geq 0$ such that, for all $\lambda_0 \in \mathbb{R}^+$, we have

$$\left\| \frac{1}{\lambda_0^{(d-2)/2}} u(t_n, \frac{x}{\lambda_0}) - u(t_{n'}, x) \right\|_{\dot{H}^1} \geq \eta_0, \quad \text{for } n \neq n'. \quad (4.30)$$

After passing to a subsequence, we assume that $t_n \rightarrow \bar{t} \in [0, T_+(u_0)]$. By taking $\lambda_0 = 1$ in (4.30) and the continuity of the flow $u(t)$ in \dot{H}^1 , we must have

$$\bar{t} = T_+(u_0).$$

In addition, from Proposition 2.1, we also have

$$\|e^{it\Delta} u(t_n)\|_{S(0, +\infty)} \geq \delta. \quad (4.31)$$

Applying Lemma 4.1 to $v_{0,n} = u(t_n)$, we have

$$u(t_n, x) = \sum_{j=1}^J \frac{1}{\lambda_{j,n}^{(d-2)/2}} V_j^l \left(-\frac{t_{j,n}}{\lambda_{j,n}^2}, \frac{x}{\lambda_{j,n}} \right) + w_n^J(x),$$

with

$$\begin{aligned}\|\nabla u(t_n)\|_{L^2}^2 &= \sum_{j=1}^J \|\nabla V_{0,j}\|_{L^2}^2 + \|\nabla w_n^J\|_{L^2}^2 + o_n(1), \\ E(u(t_n)) &= \sum_{j=1}^J E(V_j^l(-\frac{t_{j,n}}{\lambda_{j,n}^2})) + E(w_n^J) + o_n(1).\end{aligned}$$

Arguing as in the proof of Proposition 4.1, we see that

$$\liminf_{n \rightarrow \infty} E(V_1^l(-\frac{t_{1,n}}{\lambda_{1,n}^2})) = E_c,$$

this implies that $J = 1$, i. e.

$$u(t_n) = \frac{1}{\lambda_{1,n}^{(d-2)/2}} V_1^l\left(-\frac{t_{1,n}}{\lambda_{1,n}^2}, \frac{x}{\lambda_{1,n}}\right) + w_n, \quad \lim_{n \rightarrow +\infty} \|w_n\|_{\dot{H}^1} \rightarrow 0. \quad (4.32)$$

The next step is to show that

$$s_n = -\frac{t_{1,n}}{\lambda_{1,n}^2} \text{ must be bounded.}$$

Notice that we have

$$e^{it\Delta} u(t_n) = \frac{1}{\lambda_{1,n}^{(d-2)/2}} V_1^l\left(\frac{t-t_{1,n}}{\lambda_{1,n}^2}, \frac{x}{\lambda_{1,n}}\right) + e^{it\Delta} w_n,$$

with $\|e^{it\Delta} w_n\|_{X(\mathbb{R})} < \frac{\delta}{2}$ for n sufficiently large.

Assume that $\frac{t_{1,n}}{\lambda_{1,n}^2} \leq -C_0$ for n large, C_0 a large positive constant. Since

$$\left\| \frac{1}{\lambda_{1,n}^{(d-2)/2}} V_1^l\left(\frac{t-t_{1,n}}{\lambda_{1,n}^2}, \frac{x}{\lambda_{1,n}}\right) \right\|_{X(0,+\infty)} \leq \|V_1^l\|_{X(C_0, \infty)} < \frac{\delta}{2}$$

for C_0 large, we get for n large

$$\|e^{it\Delta} u(t_n)\|_{X(0,+\infty)} < \delta,$$

which is a contradiction to (4.31).

On the other hand, if $\frac{t_{1,n}}{\lambda_{1,n}^2} \geq C_0$ for n large, we have

$$\left\| \frac{1}{\lambda_{1,n}^{(d-2)/2}} V_1^l\left(\frac{t-t_{1,n}}{\lambda_{1,n}^2}, \frac{x}{\lambda_{1,n}}\right) \right\|_{X(-\infty,0)} \leq \|V_1^l\|_{X(-\infty, -C_0)} < \frac{\delta}{2}$$

for C_0 large. Hence,

$$\|e^{it\Delta} u(t_n)\|_{X(-\infty, t_n)} \leq \delta$$

for n large, Proposition 2.1 now gives

$$\|u\|_{X(-\infty, t_n)} \leq 2\delta.$$

Since $t_n \rightarrow \bar{t} = T_+(u_0)$, we also obtain a contradiction.

Hence

$$\left| -\frac{t_{1,n}}{\lambda_{1,n}^2} \right| \leq C_0,$$

after passing to a subsequence, we can assume that

$$\frac{t_{1,n}}{\lambda_{1,n}^2} \rightarrow t_0 \in (-\infty, +\infty).$$

On the other hand, by (4.30) and (4.32), we obtain that for $n \neq n'$ large,

$$\left\| \frac{1}{\lambda_0^{(d-2)/2}} \frac{1}{\lambda_{1,n}^{(d-2)/2}} V_1^l \left(-\frac{t_{1,n}}{\lambda_{1,n}^2}, \frac{x}{\lambda_0} \right) - \frac{1}{\lambda_{1,n'}^{(d-2)/2}} V_1^l \left(-\frac{t_{1,n'}}{\lambda_{1,n'}^2}, \frac{x}{\lambda_{1,n'}} \right) \right\|_{\dot{H}^1} \geq \frac{\eta_0}{2},$$

or

$$\left\| \left(\frac{\lambda_{1,n'}}{\lambda_0 \lambda_{1,n}} \right)^{(d-2)/2} V_1^l \left(-\frac{t_{1,n}}{\lambda_{1,n}^2}, \frac{\lambda_{1,n'}}{\lambda_0 \lambda_{1,n}} y \right) - V_1^l \left(-\frac{t_{1,n'}}{\lambda_{1,n'}^2}, y \right) \right\|_{\dot{H}^1} \geq \frac{\eta_0}{2}.$$

Letting

$$\lambda_0 = \frac{\lambda_{1,n'}}{\lambda_{1,n}},$$

we will obtain a contradiction because of the continuity of the linear flow $V_1^l(t, x)$ in \dot{H}^1 and

$$-\frac{t_{1,n}}{\lambda_{1,n}^2} \rightarrow t_0, \quad \text{and} \quad -\frac{t_{1,n'}}{\lambda_{1,n'}^2} \rightarrow t_0.$$

This completes the proof.

5 Rigidity theorem

In this section, we will prove main theorem.

Theorem 5.1. *Assume that $u_0 \in \dot{H}^1$ is radial and satisfies*

$$E(u_0) < E(W), \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}.$$

Let u be the solution of (2.1) with maximal interval of existence $(-T_-(u_0), T_+(u_0))$. Assume that there exists $\lambda(t) > 0$, for $t \in [0, T_+(u_0))$, with the property that

$$K = \left\{ v(t, x) = \frac{1}{\lambda(t)^{\frac{d-2}{2}}} u(t, \frac{x}{\lambda(t)}) , t \in [0, T_+(u_0)) \right\}$$

is precompact in \dot{H}^1 . Then $T_+(u_0) = +\infty$, $u_0 \equiv 0$.

We start out with a special case of the strengthened form of Theorem 5.1

Proposition 5.1. *Assume that $u, v, \lambda(t)$ are as in Theorem 5.1, and that $\lambda(t) \geq A_0 > 0$. Then the conclusion of Theorem 5.1 holds.*

First we collect some useful facts:

Lemma 5.1. *Let u, v be as in Theorem 5.1.*

(1) *Let $\delta_0 > 0$ be such that $E(u_0) \leq (1 - \delta_0)E(W)$. Then there exists $\bar{\delta} > 0$ such that for all $t \in [0, T_+(u_0))$, we have*

$$\begin{aligned} \int |\nabla u(t)|^2 dx &\leq (1 - \bar{\delta}) \int |\nabla W|^2 dx, \\ \int |\nabla u(t, x)|^2 dx - \int \int \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dx dy &\geq \frac{\bar{\delta}}{2} \int |\nabla u|^2 dx, \\ \int |\nabla u(t)|^2 dx &\approx E(u(t)) = E(u_0) \approx \int |\nabla u_0|^2 dx. \end{aligned} \quad (5.1)$$

(2) *For all $t \in [0, T_+(u_0))$, we have*

$$\|v(t, x)\|_{L^{2^*}}^2 \leq C_1 \int |\nabla v(t, x)|^2 dx \leq C_2 \int |\nabla W(x)|^2 dx.$$

(3) *For each ϵ , there exists $R(\epsilon) > 0$, such that for $t \in [0, T_+(u_0))$, we have*

$$\int_{|x|>R(\epsilon)} |\nabla v(t, x)|^2 + |v(t, x)|^{2^*} + \frac{|v(t, x)|^2}{|x|^2} dx + \iint_{\Omega} \frac{|v(t, x)|^2 |v(t, y)|^2}{|x - y|^4} dx dy \leq \epsilon, \quad (5.2)$$

where

$$\Omega = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |x| > R(\epsilon)\} \cup \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |y| > R(\epsilon)\}.$$

Proof: From the property of K , we can easily verify them.

Proof of Proposition 5.1: We split the proof into two cases, the finite time blowup for u and the infinite time of existence for u .

Case 1: $T_+(u_0) < +\infty$. We claim that

$$\lambda(t) \rightarrow \infty \quad \text{as } t \rightarrow T_+(u_0).$$

Its proof is analogue to the proof of Proposition 5.3 in [12] and Corollary 1.10 in [15]. If not, there exists $t_i \nearrow T_+(u_0)$ with $\lambda(t_i) \rightarrow \lambda_0 \in [A_0, +\infty)$.

Let

$$v_i(x) = \frac{1}{\lambda(t_i)^{\frac{d-2}{2}}} u\left(t_i, \frac{x}{\lambda(t_i)}\right)$$

from the compactness of \overline{K} , there exists $v(x) \in \dot{H}^1$ with

$$v_i \rightarrow v \quad \text{in } \dot{H}^1,$$

Thus, we have

$$u(t_i, x) = \lambda(t_i)^{\frac{d-2}{2}} v_i(\lambda(t_i)x) \rightarrow \lambda_0^{\frac{d-2}{2}} v(\lambda_0 x) \quad \text{in } \dot{H}^1,$$

Let $h(t, x)$ be the solution of (2.1) with data $\lambda_0^{\frac{d-2}{2}} v(\lambda_0 x)$ at time $T_+(u_0)$ in an interval $(T_+(u_0) - \delta, T_+(u_0) + \delta)$ with

$$\|h\|_{X((T_+(u_0) - \delta, T_+(u_0) + \delta))} < \infty.$$

Let $h_i(t, x)$ be the solution with data at $T_+(u_0)$ equal to $u(t_i, x)$. Then the local well-posedness theory and Remark 2.6 guarantee that

$$\sup_i \|h_i(t, x)\|_{X((T_+(u_0) - \frac{\delta}{2}, T_+(u_0) + \frac{\delta}{2}))} < \infty.$$

Since $h_i(t, x) = u(t + t_i - T_+(u_0), x)$ and $T_+(u_0) < \infty$, It gives a contradiction with Proposition 2.2.

Now let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be radial, and

$$\varphi(x) = \begin{cases} 1, & \text{for } |x| \leq 1; \\ 0, & \text{for } |x| \geq 2 \end{cases}$$

Set

$$\varphi_R(x) = \varphi\left(\frac{x}{R}\right).$$

Define

$$y_R(t) = \int \varphi_R(x) |u(t, x)|^2 dx, \quad t \in [0, T_+(u_0)).$$

From Lemma 5.1 and Lemma 3.2, we then have

$$\begin{aligned} |y'_R(t)| &\lesssim \int |u(t, x) \nabla u(t, x) \nabla (\varphi_R(x))| dx \\ &\lesssim \|\nabla u(t)\|_{L^2} \left\| \frac{u(t, x)}{|x|} \right\|_{L^2} \lesssim \|\nabla W(x)\|_{L^2}^2. \end{aligned} \tag{5.3}$$

On the other hand, we also have

$$\forall R > 0, \quad \int_{|x| < R} |u(t, x)|^2 dx \rightarrow 0 \quad \text{as } t \rightarrow T_+(u_0). \tag{5.4}$$

Indeed, since $u(t, x) = \lambda(t)^{\frac{d-2}{2}} v(t, \lambda(t)x)$, we have from Hölder's inequality

$$\begin{aligned} \int_{|x| < R} |u(t, x)|^2 dx &= \lambda(t)^{-2} \int_{|y| < \lambda(t)R} |v(t, y)|^2 dy \\ &= \lambda(t)^{-2} \int_{|y| < \epsilon\lambda(t)R} |v(t, y)|^2 dy + \lambda(t)^{-2} \int_{\epsilon\lambda(t)R \leq |y| \leq \lambda(t)R} |v(t, y)|^2 dy \\ &\leq \lambda(t)^{-2} (\epsilon\lambda(t)R)^2 \|v(t, x)\|_{L^{2^*}}^2 + \lambda(t)^{-2} (\lambda(t)R)^2 \|v(t, x)\|_{L^{2^*}(|x| \geq \epsilon\lambda(t)R)}^2 \\ &= C_3 (\epsilon R)^2 \int |\nabla W(x)|^2 dx + R^2 \|v(t, x)\|_{L^{2^*}(|x| \geq \epsilon\lambda(t)R)}^2. \end{aligned}$$

The first term is small with ϵ . Lemma 5.1 implies that the second term tends to 0 as t tends to $T_+(u_0)$.

From (5.4), we have

$$y_R(t) \rightarrow 0 \quad \text{as } t \rightarrow T_+(u_0). \quad (5.5)$$

From (5.3) and (5.5), we have

$$\begin{aligned} y_R(0) &\leq y_R(T_+(u_0)) + C T_+(u_0) \int |\nabla W(x)|^2 dx \\ &= C T_+(u_0) \int |\nabla W(x)|^2 dx \end{aligned}$$

where $y_R(T_+(u_0))$ denotes $\lim_{t \nearrow T_+(u_0)} y_R(t)$.

Thus, letting $R \rightarrow +\infty$, we obtain

$$u_0 \in L^2(\mathbb{R}^d).$$

Arguing as before, we have

$$|y_R(t)| = |y_R(t) - y_R(T_+(u_0))| \leq C (T_+(u_0) - t) \int |\nabla W(x)|^2 dx.$$

Letting $R \rightarrow +\infty$, we have

$$\|u(t)\|_{L^2}^2 \leq C (T_+(u_0) - t) \int |\nabla W(x)|^2 dx.$$

By the conservation of mass, this implies

$$u_0 \equiv 0$$

which is in contradiction with $T_+(u_0) < +\infty$.

Case 2: $T_+(u_0) = +\infty$. On one hand, from $u(t, x) = \lambda(t)^{\frac{d-2}{2}} v(t, \lambda(t)x)$ and Lemma 5.1, we have for each $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that

$$\int_{|x| > R(\epsilon)} \frac{|u(t, x)|^2}{|x|^2} dx + \int_{|x| > R(\epsilon)} |\nabla u(t, x)|^2 dx + \iint_{\Omega} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x-y|^4} dx dy \leq \epsilon, \quad (5.6)$$

where

$$\Omega = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |x| > R(\epsilon)\} \cup \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |y| > R(\epsilon)\}.$$

On the other hand, from Lemma 5.1, and (5.6), there exists R such that, for all $t \in [0, +\infty)$

$$8 \int_{|x| \leq R} |\nabla u(t, x)|^2 dx - 8 \iint_{\Omega_1} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dxdy \geq C_{\delta_0} \int |\nabla u_0(x)|^2 dx, \quad (5.7)$$

where

$$\Omega_1 = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |x| \leq R, |y| \leq R\}.$$

Now let $\varphi \in C_0^\infty(\mathbb{R}^d)$ be radial, and

$$\varphi(x) = \begin{cases} |x|^2, & \text{for } |x| \leq 1; \\ 0, & \text{for } |x| \geq 2, \end{cases}$$

Set

$$\varphi_R(x) = R^2 \varphi\left(\frac{x}{R}\right).$$

Define

$$z_R(t) = \int \varphi_R(x) |u(t, x)|^2 dx, \quad t \in [0, T_+(u_0)).$$

We then have

$$\begin{aligned} |z'_R(t)| &\leq CR^2 \int |\nabla u_0|^2 dx, \quad \text{for } t > 0, \\ z''_R(t) &\geq C_{\delta_0} \int |\nabla u_0|^2 dx, \quad \text{for } R \text{ large enough, } t > 0. \end{aligned} \quad (5.8)$$

In fact, from Lemma 5.1 and Lemma 3.2, we have

$$\begin{aligned} |z'_R(t)| &\leq 2R \int \left| \bar{u}(t, x) \nabla u(t, x) \nabla \varphi\left(\frac{x}{R}\right) \right| dx \\ &\leq CR \int_{|x| \leq 2R} |u| |\nabla u| dx \leq CR^2 \|\nabla u(t, x)\|_{L^2} \left\| \frac{|u|}{|x|} \right\|_{L^2} \leq CR^2 \int |\nabla u_0|^2 dx. \end{aligned}$$

On the other hand, from Lemma 3.2, (5.6) and (5.7), we have for sufficiently large R

$$\begin{aligned} z''_R(t) &= - \int \Delta \Delta \varphi\left(\frac{x}{R}\right) \frac{|u|^2}{R^2} dx + 4 \operatorname{Re} \int \varphi_{jk} \bar{u}_j u_k dx \\ &\quad - 4 \operatorname{Re} \iint (a_j(x) - a_j(y)) \frac{x_j - y_j}{|x - y|^6} |u(t, x)|^2 |u(t, y)|^2 dxdy \\ &\approx 8 \int_{|x| \leq R} |\nabla u(t, x)|^2 dx - 8 \iint_{\Omega_1} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dxdy \\ &\quad + O\left(\int_{|x| \approx R} \frac{|u(t, x)|^2}{R^2} dx + \int_{|x| \approx R} |\nabla u(t, x)|^2 dx + \iint_{\Omega_2} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dxdy \right) \\ &\geq C_{\delta_0} \int |\nabla u_0|^2 dx, \end{aligned}$$

where

$$\begin{aligned}\Omega_1 &= \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |x| \leq R, |y| \leq R\}; \\ \Omega_2 &= \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |x| \sim R\} \cup \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; |y| \sim R\}.\end{aligned}$$

From (5.8), we have

$$C_{\delta_0} t \int |\nabla u_0|^2 dx \leq |z'_R(t) - z'_R(0)| \leq 2CR^2 \int |\nabla u_0|^2 dx.$$

We have a contradiction for t large unless $u_0 \equiv 0$.

Proof of Theorem 5.1: It is analogue to the proof of [12], [21]. Assume that $u_0 \not\equiv 0$, then

$$\int |\nabla u_0|^2 dx > 0. \quad (5.9)$$

From Lemma 5.1, we have

$$E(u_0) \geq C_{\delta_0} \int |\nabla u_0|^2 dx > 0.$$

Because of Proposition 5.1, we only need to consider the case where there exists $\{t_n\}_{n=1}^{+\infty}$, $t_n \geq 0$, such that

$$\lambda(t_n) \rightarrow 0.$$

We claim that

$$t_n \rightarrow T_+(u_0).$$

Indeed, if $t_n \rightarrow t_0 \in [0, T_+(u_0))$, then we have for all $R > 0$

$$\begin{aligned}\int_{|x|>R} |v(t_n, x)|^{2^*} dx &= \int_{|x|>R} \left| \frac{1}{\lambda(t_n)^{\frac{d-2}{2}}} u(t_n, \frac{x}{\lambda(t_n)}) \right|^{2^*} dx \\ &= \int_{|x|>\frac{R}{\lambda(t_n)}} |u(t_n, x)|^{2^*} dx.\end{aligned}$$

Because of $u \in C_t^0([0, T_+(u_0)); \dot{H}^1)$, we have

$$\int_{|x|>R} |v(t_0, x)|^{2^*} dx = 0, \quad \forall R > 0.$$

It is in contradiction with the fact that

$$\int |\nabla v(t_0, x)|^2 dx = \int |\nabla u(t_0, x)|^2 dx > 0.$$

Now after possibly redefining $\{t_n\}_{n=1}^{+\infty}$, we can assume that

$$\lambda(t_n) \leq 2 \inf_{t \in [0, t_n]} \lambda(t). \quad (5.10)$$

From the hypothesis, we have

$$w_n(x) = \frac{1}{\lambda(t_n)^{\frac{d-2}{2}}} u(t_n, \frac{x}{\lambda(t_n)}) \rightarrow w_0 \quad \text{in } \dot{H}^1.$$

By Proposition 3.1, we have

$$\begin{aligned} \int |\nabla w_n(x)|^2 dx &= \int |\nabla u(t_n, x)|^2 dx < (1 - \bar{\delta}) \int |\nabla W(x)|^2 dx, \\ E(w_n) &= E(u(t_n)) = E(u_0) < E(W). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \int |\nabla w_0|^2 dx &\leq (1 - \bar{\delta}) \int |\nabla W(x)|^2 dx \\ 0 &< E(w_0) = E(u_0) < E(W). \end{aligned}$$

Thus $w_0 \not\equiv 0$. Let us now consider solutions $w_n(\tau, x), w_0(\tau, x)$ of (2.1) with data $w_n(x), w_0(x)$ at $\tau = 0$, defined in maximal intervals $\tau \in (-T_-(w_n), 0]$ and $\tau \in (-T_-(w_0), 0]$, respectively.

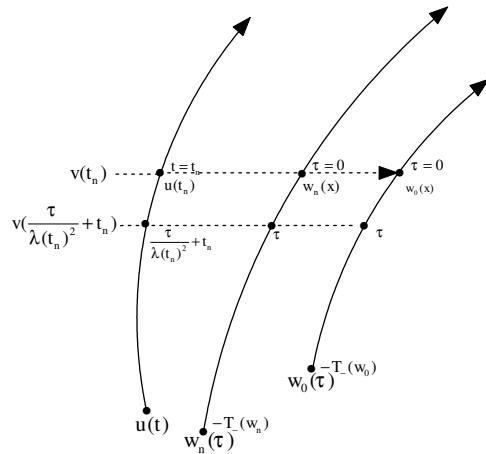


Figure 2: A description of the normalization on $\lambda(t)$.

Since $w_n(x) \rightarrow w_0(x)$ in \dot{H}^1 , we have from Remark 2.6 that

$$\begin{aligned} \lim_{n \rightarrow +\infty} T_-(w_n) &\geq T_-(w_0), \\ w_n(\tau, x) &\rightarrow w_0(\tau, x) \quad \text{in } \dot{H}^1, \quad \forall \tau \in (-T_-(w_0), 0]. \end{aligned} \tag{5.11}$$

By the uniqueness of solution of (2.1), we have

$$w_n(\tau, x) = \frac{1}{\lambda(t_n)^{\frac{d-2}{2}}} u\left(\frac{\tau}{\lambda(t_n)^2} + t_n, \frac{x}{\lambda(t_n)}\right), \quad \text{for } \frac{\tau}{\lambda(t_n)^2} + t_n \geq 0.$$

Now we claim that

$$\lim_{n \rightarrow +\infty} t_n \lambda(t_n)^2 \geq T_-(w_0). \tag{5.12}$$

Indeed, if not, then $\lim_{n \rightarrow +\infty} t_n \lambda(t_n)^2 \rightarrow \tau_0 < T_-(w_0)$, from (5.11), we have as $n \rightarrow +\infty$

$$w_n(-t_n \lambda(t_n)^2, x) = \frac{1}{\lambda(t_n)^{\frac{d-2}{2}}} u_0\left(\frac{x}{\lambda(t_n)}\right) \rightarrow w_0(-\tau_0, x) \quad \text{in } \dot{H}^1.$$

Note that from $\lambda(t_n) \rightarrow 0$, we have as $n \rightarrow +\infty$

$$\frac{1}{\lambda(t_n)^{\frac{d-2}{2}}} u_0\left(\frac{x}{\lambda(t_n)}\right) \rightarrow 0 \quad \text{in } \dot{H}^1,$$

thus we obtain that $w_0(-\tau_0) \equiv 0$, which yields a contradiction.

From (5.12), we have that for fixed $\tau \in (-T_-(w_0), 0]$ and sufficiently large n ,

$$0 \leq \frac{\tau}{\lambda(t_n)^2} + t_n \leq t_n,$$

$v(\frac{\tau}{\lambda(t_n)^2} + t_n, x)$, $\lambda(\frac{\tau}{\lambda(t_n)^2} + t_n)$ are defined and we have

$$\begin{aligned} v\left(\frac{\tau}{\lambda(t_n)^2} + t_n, x\right) &= \frac{1}{\lambda\left(\frac{\tau}{\lambda(t_n)^2} + t_n\right)^{\frac{d-2}{2}}} u\left(\frac{\tau}{\lambda(t_n)^2} + t_n, \frac{x}{\lambda\left(\frac{\tau}{\lambda(t_n)^2} + t_n\right)}\right) \\ &= \frac{1}{\tilde{\lambda}_n(\tau)^{\frac{d-2}{2}}} w_n\left(\tau, \frac{x}{\tilde{\lambda}_n(\tau)}\right), \end{aligned}$$

where

$$\tilde{\lambda}_n(\tau) = \frac{\lambda\left(\frac{\tau}{\lambda(t_n)^2} + t_n\right)}{\lambda(t_n)} \geq \frac{1}{2}$$

because of the fact (5.10). After passing to a subsequence, we can assume that

$$\tilde{\lambda}_n(\tau) \rightarrow \tilde{\lambda}_0(\tau) \in [\frac{1}{2}, +\infty].$$

Hence, we have

$$v\left(\frac{\tau}{\lambda(t_n)^2} + t_n, x\right) \rightarrow \frac{1}{\tilde{\lambda}_0(\tau)^{\frac{d-2}{2}}} w_0\left(\tau, \frac{x}{\tilde{\lambda}_0(\tau)}\right) = v_0(\tau, x) \in \overline{K}.$$

Now we claim that

$$\tilde{\lambda}_0(\tau) < +\infty.$$

If not, from

$$\frac{1}{\tilde{\lambda}_n(\tau)^{\frac{d-2}{2}}} w_n\left(\tau, \frac{x}{\tilde{\lambda}_n(\tau)}\right) \rightarrow \frac{1}{\lambda_0(\tau)^{\frac{d-2}{2}}} w_0\left(\tau, \frac{x}{\lambda_0(\tau)}\right) = v_0(\tau, x),$$

we have

$$w_0(\tau) = 0,$$

which yields a contradiction.

So far, $w_0(\tau)$, $v_0(\tau)$ and $\tilde{\lambda}_0(\tau)$ satisfy the conditions of Proposition 5.1, we obtain that

$$w_0 \equiv 0,$$

which yields a contradiction. This completes the proof.

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